THE DILATION PROPERTY OF MODULATION SPACES AND THEIR INCLUSION RELATION WITH BESOV SPACES

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Abstract. We consider the dilation property of the modulation spaces $M^{p,q}$. Let $D_\lambda : f(t) \mapsto f(\lambda t)$ be the dilation operator, and we consider the behavior of the operator norm $\|D_\lambda\|_{M^{p,q} \to M^{p,q}}$ with respect to λ. Our result determines the best order for it, and as an application, we establish the optimality of the inclusion relation between the modulation spaces and Besov space, which was proved by Toft [9].

1. Introduction

The modulation spaces $M^{p,q}$ were first introduced by Feichtinger in [1] and [2]. The exact definition will be given in the next section, but the main idea is to consider the decaying property of a function with respect to the space variable and the variable of its Fourier transform simultaneously. That is exactly the heart of the matter of the time-frequency analysis which is originated in signal analysis or quantum mechanics.

Based on a similar idea, Sjöstrand [8] independently introduced a symbol class which assures the L^2-boundedness of corresponding pseudo-differential operators. In the last decade, the theory of the modulation spaces has been developed, and its usefulness for the theory of pseudo-differential operators is getting realized gradually. Nowadays Sjöstrand's symbol class is recognized as a special case of the modulation spaces by Gröchenig [5]. Gröchenig and Heil [6] also used the modulation spaces, as a powerful tool, to show trace-class results for pseudo-differential operators. Consult Gröchenig [4] for further and detailed history of this research fields.

Now we are in a situation to start showing fundamental properties of the modulation spaces, in order to apply them for many other problems. Actually in Toft's recent work [9], he investigated the mapping property of convolutions, and showed Young-type results for the modulation space. As an application, he showed an inclusion relation between the modulation spaces and Besov spaces. We remark that Besov spaces are used in various problems of partial differential equations, and his result will help us to understand how they are translated into the terminology of the modulation spaces.

Among many other important properties to be shown, we focus on the dilation property of the modulation spaces in this article. Since $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, we have easily $\|f_\lambda\|_{M^{2,2}} = \lambda^{-n/2}\|f\|_{M^{2,2}}$ by the change of variables $t \mapsto \lambda^{-1}t$, where $f_\lambda(t) = f(\lambda t)$ and $t \in \mathbb{R}^n$. But it is not clear how $\|f_\lambda\|_{M^{p,q}}$ behaves like with respect to λ except for the case $(p, q) = (2, 2)$. Our objective is to draw the complete picture of the best order of λ for every pair of (p, q) (Theorem 1.1).

We can expect various kinds of applications of this consideration. In fact, this kind of dilation property is frequently used in the "scaling argument", which is a popular tool to know the best possible order of the conditions in problems of partial differential

2000 *Mathematics Subject Classification.* 42B35.

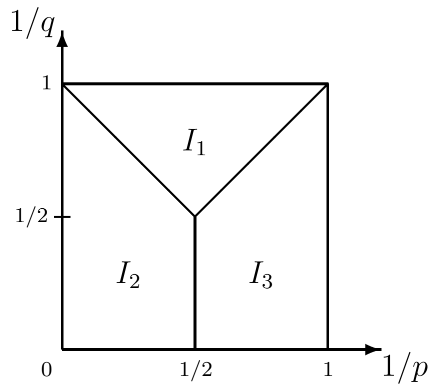
Key words and phrases. Modulation spaces, Besov spaces, dilation, inclusion.

equations. Actually, in this article, we also show the best possibility of Toft's inclusion relation mentioned above, as a side product of the main argument (Theorem 1.2).

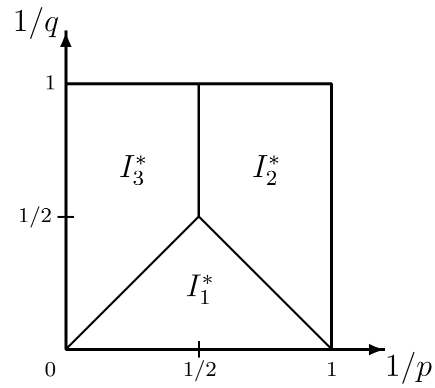
In order to state our main results, we introduce several indexes. For $1 \le p \le \infty$, we denote the conjugate exponent of p by p' (that is, $1/p + 1/p' = 1$). We define subsets of $(1/p, 1/q) \in [0,1] \times [0,1]$ in the following way:

$$\begin{aligned} I_1 &: \max(1/p, 1/p') \le 1/q, & I_1^* &: \min(1/p, 1/p') \ge 1/q, \\ I_2 &: \max(1/q, 1/2) \le 1/p', & I_2^* &: \min(1/q, 1/2) \ge 1/p', \\ I_3 &: \max(1/q, 1/2) \le 1/p, & I_3^* &: \min(1/q, 1/2) \ge 1/p. \end{aligned}$$

See the following figure:



$0 < \lambda \le 1$



$\lambda \ge 1$

In [9], Toft introduced the indexes

$$\nu_1(p,q) = \max\{0, 1/q - \min(1/p, 1/p')\},$$
$$\nu_2(p,q) = \min\{0, 1/q - \max(1/p, 1/p')\}.$$

Note that

$$\nu_1(p,q) = \begin{cases} 0 & \text{if} \quad (1/p, 1/q) \in I_1^*, \\ 1/p + 1/q - 1 & \text{if} \quad (1/p, 1/q) \in I_2^*, \\ -1/p + 1/q & \text{if} \quad (1/p, 1/q) \in I_3^*, \end{cases}$$

and

$$\nu_2(p,q) = \begin{cases} 0 & \text{if} \quad (1/p, 1/q) \in I_1, \\ 1/p + 1/q - 1 & \text{if} \quad (1/p, 1/q) \in I_2, \\ -1/p + 1/q & \text{if} \quad (1/p, 1/q) \in I_3. \end{cases}$$

We also introduce the indexes

$$\mu_1(p,q) = \nu_1(p,q) - 1/p, \qquad \mu_2(p,q) = \nu_2(p,q) - 1/p.$$

Then we have

$$\mu_1(p,q) = \begin{cases} -1/p & \text{if} \quad (1/p, 1/q) \in I_1^*, \\ 1/q - 1 & \text{if} \quad (1/p, 1/q) \in I_2^*, \\ -2/p + 1/q & \text{if} \quad (1/p, 1/q) \in I_3^*, \end{cases}$$

and

$$\mu_2(p,q) = \begin{cases} -1/p & \text{if} \quad (1/p, 1/q) \in I_1, \\ 1/q - 1 & \text{if} \quad (1/p, 1/q) \in I_2, \\ -2/p + 1/q & \text{if} \quad (1/p, 1/q) \in I_3. \end{cases}$$

Our first main result is on the dilation property of the modulation spaces. For a function (or tempered distribution) f on $\mathbb{R}^n$ and $\lambda > 0$, we use the notation f_λ which is defined by $f_\lambda(t) = f(\lambda t)$, $t \in \mathbb{R}^n$.

Theorem 1.1. *Let $1 \le p, q \le \infty$. Then the following are true*:

(1) *There exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{p,q}} \le C\lambda^{n\mu_1(p,q)} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } \lambda \ge 1.$$

Conversely, if there exist constants $C > 0$ and $\alpha \in \mathbb{R}$ such that

$$\|f_\lambda\|_{M^{p,q}} \le C\lambda^{\alpha} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } \lambda \ge 1,$$

then $\alpha \ge n\mu_1(p,q)$.

(2) *There exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{p,q}} \le C\lambda^{n\mu_2(p,q)} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \le 1.$$

Conversely, if there exist constants $C > 0$ and $\beta \in \mathbb{R}$ such that

$$\|f_\lambda\|_{M^{p,q}} \le C\lambda^{\beta} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \le 1,$$

then $\beta \le n\mu_2(p,q)$.

Since the Gauss function $\varphi(t) = e^{-|t|^2}$ does not change its form under the Fourier transformation, the modulation norm of it can have a "good" property. In this sense, it is reasonable to believe that the Gauss function $f = \varphi$ attains the critical order of $\|f_\lambda\|_{M^{p,q}}$ with respect to λ. But it is not true because $\|\varphi_\lambda\|_{M^{p,q}} \sim \lambda^{n(1/q-1)}$ in the case $\lambda \ge 1$ and $\|\varphi_\lambda\|_{M^{p,q}} \sim \lambda^{-n/p}$ in the case $0 < \lambda \le 1$ (see Lemma 2.1). Theorem 1.1 says that they are not critical orders for every pair of (p,q).

It should be pointed out here that the behavior of $\|f_\lambda\|_{M^{p,q}}$ with respect to λ might depend on the choice of $f \in M^{p,q}(\mathbb{R}^n)$. In fact, $f(t) = \sum_{k\in\mathbb{Z}^n} e^{ik\cdot t}\, \psi(t-k)$, where ψ is an appropriate Schwartz function, has the property $\|f_\lambda\|_{M^{p,\infty}} \sim \lambda^{-2n/p}$ $(0 < \lambda \le 1)$ in the case $1 \le p \le 2$ (Lemma 3.9), while the Gauss function has the different behavior $\|\varphi_\lambda\|_{M^{p,\infty}} \sim \lambda^{-n/p}$ $(0 < \lambda \le 1)$ as mentioned above. On the other hand, the L^p-norm never has such a property since $\|f_\lambda\|_{L^p} = \lambda^{-n/p}\|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^n)$. That is one of great differences between the modulation spaces and L^p-spaces.

Our second main result is on the optimality of the inclusion relation between the modulation spaces and Besov spaces. In [9, Theorem 3.1], Toft proved the inclusions

$$B^{p,q}_{n\nu_1(p,q)}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n) \hookrightarrow B^{p,q}_{n\nu_2(p,q)}(\mathbb{R}^n)$$

for $1 \le p, q \le \infty$. He also remarked that the left inclusion is optimal in the case $1 \le p = q \le 2$, that is, if $B^{p,p}_{s_1}(\mathbb{R}^n) \hookrightarrow M^{p,p}(\mathbb{R}^n)$ then $s_1 \ge n\nu_1(p,p)$. The same is true for the right inclusion in the case $2 \le p = q \le \infty$, that is, if $M^{p,p}(\mathbb{R}^n) \hookrightarrow B^{p,p}_{s_2}(\mathbb{R}^n)$ then $s_2 \le n\nu_2(p,p)$ ([9, Remark 3.11]). The next theorem says that Toft's inclusion result is optimal in the above meaning for every pair of (p,q).

Theorem 1.2. *Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. Then the following are true*:

(1) *If $B^{p,q}_s(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n)$, then $s \ge n\nu_1(p,q)$.*
(2) *If $M^{p,q}(\mathbb{R}^n) \hookrightarrow B^{p,q}_s(\mathbb{R}^n)$ and $1 \le p, q < \infty$, then $s \le n\nu_2(p,q)$.*

We end this introduction by explaining the plan of this article. In Section 2, we give the precise definition and basic properties of the modulation spaces and Besov spaces. In Sections 3 and 4, we prove Theorems 1.1 and 1.2 respectively.

2. Preliminaries

We introduce the modulation spaces based on Gröchenig [4]. Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\hat{f}$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\, f(x)\, dx$$

and

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, f(\xi)\, d\xi.$$

Fix a function $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ (called the window function). Then the short-time Fourier transform $V_\varphi f$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to φ is defined by

$$V_\varphi f(x,\xi) = \langle f, M_\xi T_x \varphi\rangle \quad \text{for } x, \xi \in \mathbb{R}^n,$$

where $M_\xi T_x \varphi(t) = e^{i\xi\cdot t}\varphi(t-x)$ and $\langle\cdot,\cdot\rangle$ is the inner product on $L^2(\mathbb{R}^n)$. We can express it in a form of the integral

$$V_\varphi f(x,\xi) = \int_{\mathbb{R}^n} f(t)\, \overline{\varphi(t-x)}\, e^{-i\xi\cdot t}\, dt,$$

which has actually the meaning for an appropriate function f on $\mathbb{R}^n$. We note that, for $f \in \mathcal{S}'(\mathbb{R}^n)$, $V_\varphi f$ is continuous on $\mathbb{R}^{2n}$ and $|V_\varphi f(x,\xi)| \le C(1+|x|+|\xi|)^N$ for some constants $C, N \ge 0$ ([4, Theorem 11.2.3]). Let $1 \le p, q \le \infty$. Then the modulation space $M^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M^{p,q}} = \|V_\varphi f\|_{L^{p,q}} = \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\varphi f(x,\xi)|^p\, dx \right)^{q/p} d\xi \right\}^{1/q} < \infty.$$

We note that $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ ([4, Proposition 11.3.1]) and $M^{p,q}(\mathbb{R}^n)$ is a Banach space ([4, Proposition 11.3.5]). The definition of $M^{p,q}(\mathbb{R}^n)$ is independent of the choice of the window function $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, that is, different window functions yield equivalent norms ([4, Proposition 11.3.2]).

We also introduce Besov spaces. Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. Suppose that $\eta, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\eta \subset \{\xi : |\xi| \le 2\}$, $\operatorname{supp}\psi \subset \{\xi : 1/2 \le |\xi| \le 2\}$ and $\eta(\xi) + \sum_{j=1}^{\infty} \psi(\xi/2^j) = 1$ for all $\xi \in \mathbb{R}^n$. Set $\varphi_0 = \eta$ and $\varphi_j = \psi(\cdot/2^j)$ if $j \ge 1$. Then Besov space $B_s^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_s^{p,q}} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j \hat{f}]\|_{L^p}^q \right)^{1/q} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Phi_j * f\|_{L^p}^q \right)^{1/q} < \infty,$$

where $\Phi_j = \mathcal{F}^{-1}\varphi_j$. We remark $B_s^{p,q}(\mathbb{R}^n)^* = B_{-s}^{p',q'}(\mathbb{R}^n)$ for $1 \le p, q < \infty$.

Finally, we list below the lemmas which will be used in the subsequent section. In this article, we frequently use the function $\varphi(t) = e^{-|t|^2}$ which is called the Gauss function.

Lemma 2.1 ([9, Lemma 1.8]). *Let φ be the Gauss function. Then*

$$\|V_\varphi(\varphi_\lambda)\|_{L^{p,q}} = \pi^{n(1/p+1/q+1)/2} p^{-n/2p} q^{-n/2q} 2^{n/q} \lambda^{-n/p} (1+\lambda^2)^{n(1/p+1/q-1)/2}.$$

Lemma 2.1 says that $\|\varphi_\lambda\|_{M^{p,q}} \sim \lambda^{n(1/q-1)}$ in the case $\lambda \ge 1$ and $\|\varphi_\lambda\|_{M^{p,q}} \sim \lambda^{-n/p}$ in the case $0 < \lambda \le 1$.

Lemma 2.2 ([4, Corollary 11.2.7]). *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi, \psi, \gamma \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\langle f, \varphi \rangle = \frac{1}{\langle \gamma, \psi \rangle} \int_{\mathbb{R}^{2n}} V_\psi f(x,\xi)\, \overline{V_\gamma \varphi(x,\xi)}\, dx\, d\xi \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Lemma 2.3 ([4, Lemma 11.3.3]). *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi, \psi, \gamma \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$|V_\varphi f(x,\xi)| \le \frac{1}{|\langle \gamma, \psi \rangle|} (|V_\psi f| * |V_\varphi \gamma|)(x,\xi) \quad \text{for all } x, \xi \in \mathbb{R}^n.$$

Lemma 2.4 ([4, Proposition 11.3.4 and Theorem 11.3.6]). *If $1 \le p, q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}(\mathbb{R}^n)$ and $M^{p,q}(\mathbb{R}^n)^* = M^{p',q'}(\mathbb{R}^n)$ under the duality*

$$\langle f, g \rangle_M = \frac{1}{\|\varphi\|_{L^2}^2} \int_{\mathbb{R}^{2n}} V_\varphi f(x,\xi)\, \overline{V_\varphi g(x,\xi)}\, dx\, d\xi$$

for $f \in M^{p,q}(\mathbb{R}^n)$ and $g \in M^{p',q'}(\mathbb{R}^n)$.

By Lemmas 2.2 and 2.4, if $1 < p, q \le \infty$ and $f \in M^{p,q}(\mathbb{R}^n)$ then

$$\|f\|_{M^{p,q}} = \sup |\langle f, g \rangle_M| = \sup |\langle f, g \rangle| \tag{2.1}$$

where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p',q'}} = 1$.

Lemma 2.5 ([1, Corollary 2.3]). *Let $1 \le p_1, p_2, q_1, q_2 \le \infty$ and $q_2 < \infty$. If T is a linear operator such that*

$$\|Tf\|_{M^{p_1,q_1}} \le A_1 \|f\|_{M^{p_1,q_1}} \quad \text{for all } f \in M^{p_1,q_1}(\mathbb{R}^n)$$

and

$$\|Tf\|_{M^{p_2,q_2}} \le A_2 \|f\|_{M^{p_2,q_2}} \quad \text{for all } f \in M^{p_2,q_2}(\mathbb{R}^n),$$

then

$$\|Tf\|_{M^{p,q}} \le C A_1^{1-\theta} A_2^{\theta} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n),$$

where $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, $0 \le \theta \le 1$ and C is independent of T.

Remark 2.6. Lemma 2.5 with the case $q_1 = q_2 = \infty$ is treated in [9, Remark 3.2], which says that it is true under a modification.

3. The dilation property of modulation spaces

In this section, we prove Theorem 1.1 which appeared in the introduction. We begin by preparing the following lemma:

Lemma 3.1. *Let $1 \le p, q \le \infty$. Then there exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{p,q}} \le C \lambda^{-n(1/p-1/q+1)} (1+\lambda^2)^{n/2} \|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda > 0$.

Proof. Let φ be the Gauss function, that is, $\varphi(t) = e^{-|t|^2}$. By a change of variable, we have

$$\|f_\lambda\|_{M^{p,q}} = \|V_\varphi(f_\lambda)\|_{L^{p,q}} = \lambda^{-n(1/p-1/q+1)}\|V_{\varphi_{1/\lambda}} f\|_{L^{p,q}}.$$

From Lemma 2.3 it follows that

$$\left|V_{\varphi_{1/\lambda}} f(x,\xi)\right| \le \|\varphi\|_{L^2}^{-2}(|V_\varphi f| * |V_{\varphi_{1/\lambda}}\varphi|)(x,\xi).$$

Hence, by Young's inequality and Lemma 2.1, we get

$$\begin{aligned}
\|f_\lambda\|_{M^{p,q}} &\le \lambda^{-n(1/p-1/q+1)}\|\varphi\|_{L^2}^{-2}\|V_{\varphi_{1/\lambda}}\varphi\|_{L^{1,1}}\|V_\varphi f\|_{L^{p,q}} \\
&= \lambda^{-n(1/p-1/q+1)}\|\varphi\|_{L^2}^{-2}\|V_\varphi(\varphi_{1/\lambda})\|_{L^{1,1}}\|V_\varphi f\|_{L^{p,q}} \\
&= \lambda^{-n(1/p-1/q+1)}\|\varphi\|_{L^2}^{-2}\left(\pi^{3n/2}2^n(\lambda^{-1})^{-n}(1+\lambda^{-2})^{n/2}\right)\|f\|_{M^{p,q}} \\
&= C_n\lambda^{-n(1/p-1/q+1)}(1+\lambda^2)^{n/2}\|f\|_{M^{p,q}}.
\end{aligned}$$

The proof is complete. □

We are now ready to prove Theorem 1.1 (1) with $(1/p,1/q) \in I_1^*$ and (2) with $(1/p,1/q) \in I_1$.

Proof of Theorem 1.1 (2) with $(1/p,1/q) \in I_1$. Suppose that $(1/p,1/q) \in I_1$. Then $\mu_2(p,q) = -1/p$. Let $1 \le r \le \infty$. By Lemma 3.1, we have

$$\|f_\lambda\|_{M^{r,1}} \le C\lambda^{-n/r}\|f\|_{M^{r,1}} \quad \text{for all } f \in M^{r,1}(\mathbb{R}^n) \text{ and } 0 < \lambda \le 1. \tag{3.1}$$

On the other hand, since $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, we have

$$\|f_\lambda\|_{M^{2,2}} \le C\lambda^{-n/2}\|f\|_{M^{2,2}} \quad \text{for all } f \in M^{2,2}(\mathbb{R}^n) \text{ and } 0 < \lambda \le 1. \tag{3.2}$$

Take $1 \le r \le \infty$ and $0 \le \theta \le 1$ such that $1/p = (1-\theta)/r + \theta/2$ and $1/q = (1-\theta)/1 + \theta/2$. Then, by the interpolation theorem (Lemma 2.5), (3.1) and (3.2) give

$$\|f_\lambda\|_{M^{p,q}} \le C\left(\lambda^{-n/r}\right)^{1-\theta}\left(\lambda^{-n/2}\right)^{\theta}\|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \le 1$. Since $(1-\theta)/r = 1/p+1/q-1$ and $\theta/2 = 1-1/q$, we get

$$\|f_\lambda\|_{M^{p,q}} \le C\lambda^{-n/p}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \le 1. \tag{3.3}$$

This is the first part of Theorem 1.1 (2) with $(1/p,1/q) \in I_1$.

We next prove the second part of Theorem 1.1 (2) with $(1/p,1/q) \in I_1$. Let $(1/p,1/q) \in I_1$. Assume that there exist constants $C > 0$ and $\beta \in \mathbb{R}$ such that

$$\|f_\lambda\|_{M^{p,q}} \le C\lambda^{\beta}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \le 1.$$

Let φ be the Gauss function. We note that the Gauss function belongs to $M^{p,q}(\mathbb{R}^n)$. Then, by Lemma 2.1 and our assumption, we have

$$\begin{aligned}
C_{p,q}\lambda^{-n/p} &\le C_{p,q}\lambda^{-n/p}(1+\lambda^2)^{n(1/p+1/q-1)/2} \\
&= \|V_\varphi(\varphi_\lambda)\|_{L^{p,q}} = \|\varphi_\lambda\|_{M^{p,q}} \le C\lambda^{\beta}\|\varphi\|_{M^{p,q}}
\end{aligned}$$

for all $0 < \lambda \le 1$. This is possible only if $\beta \le -n/p$. The proof is complete.

Proof of Theorem 1.1 (1) with $(1/p,1/q) \in I_1^*$. We note that $\mu_1(p,q) = -1/p$ if $(1/p,1/q) \in I_1^*$. Let $1 \le p \le \infty$ and $q \ge 2$ be such that $(1/p,1/q) \in I_1^*$. Then $(1/p',1/q') \in I_1$. We first consider the case $p \ne 1$. Since $1 < p,q \le \infty$, by duality (2.1) and Theorem 1.1 (2) with $(1/p',1/q') \in I_1$, we have

$$\|f_\lambda\|_{M^{p,q}} = \sup|\langle f_\lambda, g\rangle| = \lambda^{-n}\sup\left|\langle f, g_{1/\lambda}\rangle\right|$$

$$\begin{aligned}
&\le \lambda^{-n} \sup \|f\|_{M^{p,q}} \|g_{1/\lambda}\|_{M^{p',q'}} \\
&\le \lambda^{-n} \sup \|f\|_{M^{p,q}} \left(C(\lambda^{-1})^{-n/p'} \|g\|_{M^{p',q'}} \right) = C\lambda^{-n/p} \|f\|_{M^{p,q}}
\end{aligned}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda \ge 1$, where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p',q'}} = 1$. In the case $p = 1$, by Lemma 3.1, we see that

$$\|f_\lambda\|_{M^{1,\infty}} \le C\lambda^{-n} \|f\|_{M^{1,\infty}} \quad \text{for all } f \in M^{1,\infty}(\mathbb{R}^n) \text{ and } \lambda \ge 1.$$

Hence, we obtain the first part of Theorem 1.1 (1) with $(1/p, 1/q) \in I_1^*$.

We consider the second part of Theorem 1.1 (1) with $(1/p, 1/q) \in I_1^*$. Let $1 < p < \infty$ and $2 \le q < \infty$ be such that $(1/p, 1/q) \in I_1^*$. Assume that there exist constants $C > 0$ and $\alpha \in \mathbb{R}$ such that

$$\|g_\lambda\|_{M^{p,q}} \le C\lambda^\alpha \|g\|_{M^{p,q}} \quad \text{for all } g \in M^{p,q}(\mathbb{R}^n) \text{ and } \lambda \ge 1.$$

Then, by duality and our assumption, we have

$$\begin{aligned}
\|f_\lambda\|_{M^{p',q'}} &= \sup |\langle f_\lambda, g\rangle| = \lambda^{-n} \sup \left|\langle f, g_{1/\lambda}\rangle\right| \\
&\le \lambda^{-n} \sup \|f\|_{M^{p',q'}} \|g_{1/\lambda}\|_{M^{p,q}} \\
&\le \lambda^{-n} \sup \|f\|_{M^{p',q'}} \left(C(\lambda^{-1})^\alpha \|g\|_{M^{p,q}} \right) = C\lambda^{-n-\alpha} \|f\|_{M^{p',q'}}
\end{aligned}$$

for all $f \in M^{p',q'}(\mathbb{R}^n)$ and $0 < \lambda \le 1$, where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p,q}} = 1$. Since $(1/p', 1/q') \in I_1$, by Theorem 1.1 (2) with $(1/p', 1/q') \in I_1$, we get $-n - \alpha \le -n/p'$. This implies $\alpha \ge -n/p$.

We next consider the case $q = \infty$. Let $1 \le r \le \infty$. Assume that there exist constants $C > 0$ and $\alpha \in \mathbb{R}$ such that

$$\|f_\lambda\|_{M^{r,\infty}} \le C\lambda^\alpha \|f\|_{M^{r,\infty}} \quad \text{for all } f \in M^{r,\infty}(\mathbb{R}^n) \text{ and } \lambda \ge 1, \tag{3.4}$$

where $\alpha < -n/r$. Since $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, we have

$$\|f_\lambda\|_{M^{2,2}} \le C\lambda^{-n/2} \|f\|_{M^{2,2}} \quad \text{for all } f \in M^{2,2}(\mathbb{R}^n) \text{ and } \lambda \ge 1. \tag{3.5}$$

Then, by the interpolation theorem, (3.4) and (3.5) give

$$\|f_\lambda\|_{M^{p,q}} \le C \left(\lambda^\alpha\right)^{1-\theta} \left(\lambda^{-n/2}\right)^\theta \|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda \ge 1$, where $1/p = (1-\theta)/r + \theta/2$, $1/q = (1-\theta)/\infty + \theta/2$, and $0 < \theta < 1$. Since $0 < \theta < 1$, we note that $1 < p < \infty$, $2 < q < \infty$ and $(1/p, 1/q) \in I_1^*$. Since $p = q$ if $r = \infty$, using that $1 - \theta = r(1/p - 1/q)$ if $1 \le r < \infty$, $1 - \theta = 1 - 2/q$ if $r = \infty$ and $\theta/2 = 1/q$, we have

$$\begin{aligned}
\|f_\lambda\|_{M^{p,q}} &\le \begin{cases} C\lambda^{\alpha r(1/p-1/q)-n/q} \|f\|_{M^{p,q}}, & \text{if } \ 1 \le r < \infty \\ C\lambda^{\alpha(1-2/q)-n/q} \|f\|_{M^{p,q}}, & \text{if } \ r = \infty \end{cases} \\
&= \begin{cases} C\lambda^{\alpha r(1/p-1/q)-n/q+n/p-n/p} \|f\|_{M^{p,q}}, & \text{if } \ 1 \le r < \infty \\ C\lambda^{\alpha(1-2/q)-n/p} \|f\|_{M^{p,q}}, & \text{if } \ r = \infty \end{cases} \\
&= \begin{cases} C\lambda^{(\alpha r+n)(1/p-1/q)-n/p} \|f\|_{M^{p,q}}, & \text{if } \ 1 \le r < \infty \\ C\lambda^{\alpha(1-2/q)-n/p} \|f\|_{M^{p,q}}, & \text{if } \ r = \infty \end{cases}
\end{aligned}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda \ge 1$. However, since $(\alpha r + n)(1/p - 1/q) < 0$ if $1 \le r < \infty$ and $\alpha(1 - 2/q) < 0$ if $r = \infty$, this contradicts Theorem 1.1 (1) with $1 < p < \infty$, $2 < q < \infty$ and $(1/p, 1/q) \in I_1^*$. Therefore, α must satisfy $\alpha \ge -n/r$. The proof is complete.

Our next goal is to prove Theorem 1.1 (1) with $(1/p, 1/q) \in I_2^*$ and (2) with $(1/p, 1/q) \in I_2$.

Lemma 3.2. *Let $1 \le p, q \le \infty$ be such that $(1/p, 1/q) \in I_2^*$ and $1/p \ge 1/q$. Then there exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{p,q}} \le C\lambda^{-n(2/p-1/q)}(1+\lambda^2)^{n(1/p-1/2)}\|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda > 0$.

Proof. Let $1 \le r \le \infty$. By Lemma 3.1, we have

$$\|f_\lambda\|_{M^{1,r}} \le C\lambda^{n(1/r-2)}(1+\lambda^2)^{n/2}\|f\|_{M^{1,r}} \tag{3.6}$$

for all $f \in M^{1,r}(\mathbb{R}^n)$ and $\lambda > 0$. Let $1 \le p, q \le \infty$ be such that $(1/p, 1/q) \in I_2^*$ and $1/p \ge 1/q$. Take $1 \le r \le \infty$ and $0 \le \theta \le 1$ such that $1/p = (1-\theta)/1 + \theta/2$ and $1/q = (1-\theta)/r + \theta/2$. Then, by the interpolation theorem, (3.2), (3.5) and (3.6) give

$$\|f_\lambda\|_{M^{p,q}} \le C\left(\lambda^{n(1/r-2)}(1+\lambda^2)^{n/2}\right)^{1-\theta}\left(\lambda^{-n/2}\right)^{\theta}\|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda > 0$. Using $(1-\theta)/r = 1/p + 1/q - 1$, $1-\theta = 2/p - 1$ and $\theta/2 = -1/p + 1$, we get

$$\begin{aligned}\|f_\lambda\|_{M^{p,q}} &\le C\lambda^{n((1-\theta)/r-2(1-\theta)-\theta/2)}(1+\lambda^2)^{n(1-\theta)/2}\|f\|_{M^{p,q}}\\ &= C\lambda^{-n(2/p-1/q)}(1+\lambda^2)^{n(1/p-1/2)}\|f\|_{M^{p,q}}\end{aligned}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda > 0$. The proof is complete. $\square$

The proof of the following lemma is based on that of [10, Theorem 3].

Lemma 3.3. *Suppose that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a real-valued function satisfying $\varphi = 1$ on $[-1/2, 1/2]^n$, $\operatorname{supp}\varphi \subset [-1,1]^n$, $\varphi(t) = \varphi(-t)$ and $\sum_{k\in\mathbb{Z}^n}\varphi(t-k) = 1$ for all $t \in \mathbb{R}^n$. Then*

$$\sup_{k\in\mathbb{Z}^n}\|(M_k\Phi)*f\|_{L^2} \le \|V_\Phi f\|_{L^{2,\infty}} \le 5^n\|\Phi\|_{L^1}\sup_{k\in\mathbb{Z}^n}\|(M_k\Phi)*f\|_{L^2}$$

for all $f \in M^{2,\infty}(\mathbb{R}^n)$, where $\Phi = \mathcal{F}^{-1}\varphi$ and $M_k\Phi(t) = e^{ik\cdot t}\Phi(t)$.

Proof. Let $f \in M^{2,\infty}(\mathbb{R}^n)$. Since Φ is a real-valued function and $\Phi(t) = \Phi(-t)$ for all t, we have

$$\begin{aligned}|V_\Phi f(x,\xi)| &= \left|\int_{\mathbb{R}^n} f(t)\,\overline{\Phi(t-x)}\,e^{-i\xi\cdot t}\,dt\right|\\ &= \left|\int_{\mathbb{R}^n} f(t)\,\Phi(x-t)\,e^{i\xi\cdot(x-t)}\,dt\right| = |(M_\xi\Phi)*f(x)|.\end{aligned} \tag{3.7}$$

We first prove

$$\operatorname{ess\,sup}_{\xi\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2dx\right)^{1/2} = \sup_{\xi\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2dx\right)^{1/2}. \tag{3.8}$$

To prove (3.8), it is enough to show that $\left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2dx\right)^{1/2}$ is continuous with respect to ξ. Since $\operatorname{ess\,sup}_{\xi\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2dx\right)^{1/2} < \infty$, for each $k \in \mathbb{Z}^n$ there exists $\xi_k \in k/2 + [-1/4, 1/4]^n$ such that $\left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi_k)|^2dx\right)^{1/2} < \infty$. Then, by (3.7), we have

$$\frac{1}{(2\pi)^{n/2}}\|\varphi(\cdot-\xi_k)\,\hat{f}\|_{L^2} = \|(M_{\xi_k}\Phi)*f\|_{L^2} = \left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi_k)|^2dx\right)^{1/2} < \infty.$$

Since $k/2+[-1/4,1/4]^n \subset \xi_k+[-1/2,1/2]^n$ and $\varphi(\cdot-\xi_k)=1$ on $\xi_k+[-1/2,1/2]^n$, we see that $|\hat{f}|^2$ is integrable on $k/2+[-1/4,1/4]^n$. The arbitrariness of $k\in\mathbb{Z}^n$ gives $\hat{f}\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. By the Lebesgue dominated convergence theorem, we see that $\|\varphi(\cdot-\xi)\,\hat{f}\|_{L^2}$ is continuous with respect to ξ. Hence, $\left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2dx\right)^{1/2}$ is continuous with respect to ξ. We obtain (3.8). Then, from (3.7) and (3.8) it follows that

$$\begin{aligned}\sup_{k\in\mathbb{Z}^n}\|(M_k\Phi)*f\|_{L^2} &\le \sup_{\xi\in\mathbb{R}^n}\|(M_\xi\Phi)*f\|_{L^2} = \sup_{\xi\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2dx\right)^{1/2}\\ &= \operatorname{ess\,sup}_{\xi\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2dx\right)^{1/2} = \|V_\Phi f\|_{L^{2,\infty}}.\end{aligned}$$

We next prove $\|V_\Phi f\|_{L^{2,\infty}} \le (5^n\|\Phi\|_{L^1})\sup_{k\in\mathbb{Z}^n}\|(M_k\Phi)*f\|_{L^2}$. Let $\xi\in\mathbb{R}^n$. Since

$$\begin{aligned}M_\xi\Phi = \mathcal{F}^{-1}[\varphi(\cdot-\xi)] &= \mathcal{F}^{-1}\left[\varphi(\cdot-\xi)\left(\sum_{k\in\mathbb{Z}^n}\varphi(\cdot-k)\right)\right]\\ &= \sum_{\substack{|k_j-\xi_j|\le 2,\\ j=1,\cdots,n}}\mathcal{F}^{-1}[\varphi(\cdot-\xi)\,\varphi(\cdot-k)] = \sum_{\substack{|k_j-\xi_j|\le 2,\\ j=1,\cdots,n}}(M_\xi\Phi)*(M_k\Phi),\end{aligned}$$

by (3.7), we have

$$|V_\Phi f(x,\xi)| = |(M_\xi\Phi)*f(x)| \le \sum_{\substack{|k_j-\xi_j|\le 2,\\ j=1,\cdots,n}}|(M_\xi\Phi)*(M_k\Phi)*f(x)|.$$

Hence, by (3.8), we get

$$\begin{aligned}\|V_\Phi f\|_{L^{2,\infty}} &= \sup_{\xi\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2\,dx\right)^{1/2}\\ &\le \sup_{\xi\in\mathbb{R}^n}\sum_{\substack{|k_j-\xi_j|\le 2,\\ j=1,\cdots,n}}\|(M_\xi\Phi)*(M_k\Phi)*f\|_{L^2}\\ &\le \sup_{\xi\in\mathbb{R}^n}\sum_{\substack{|k_j-\xi_j|\le 2,\\ j=1,\cdots,n}}\|M_\xi\Phi\|_{L^1}\,\|(M_k\Phi)*f\|_{L^2}\\ &\le \|\Phi\|_{L^1}\left(\sup_{\ell\in\mathbb{Z}^n}\|(M_\ell\Phi)*f\|_{L^2}\right)\left(\sup_{\xi\in\mathbb{R}^n}\sum_{\substack{|k_j-\xi_j|\le 2,\\ j=1,\cdots,n}}1\right)\\ &\le 5^n\|\Phi\|_{L^1}\sup_{\ell\in\mathbb{Z}^n}\|(M_\ell\Phi)*f\|_{L^2}.\end{aligned}$$

The proof is complete. □

We remark that Lemma 3.1 implies

$$\|f_\lambda\|_{M^{2,\infty}} \le C\lambda^{-3n/2}\|f\|_{M^{2,\infty}} \quad \text{for all } f\in M^{2,\infty}(\mathbb{R}^n) \text{ and } 0<\lambda\le 1.$$

This is not our desired order of λ in the case $(p,q)=(2,\infty)$. But we have

Lemma 3.4. *There exists a constant $C>0$ such that*

$$\|f_\lambda\|_{M^{2,\infty}} \le C\lambda^{-n}\|f\|_{M^{2,\infty}} \quad \textit{for all } f\in M^{2,\infty}(\mathbb{R}^n) \textit{ and } 0<\lambda\le 1.$$

Proof. Let $\Phi = \mathcal{F}^{-1}\varphi$, where φ is as in Lemma 3.3. Suppose that $f \in M^{2,\infty}(\mathbb{R}^n)$. We note that $\hat{f} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ (see the proof of Lemma 3.3). Then, by Lemma 3.3, we see that

$$\begin{aligned}
\|V_\Phi(f_\lambda)\|_{M^{2,\infty}} &\le 5^n \|\Phi\|_{L^1} \sup_{k\in\mathbb{Z}^n} \|(M_k\Phi) * f_\lambda\|_{L^2} \\
&= (2\pi)^{-n/2} 5^n \|\Phi\|_{L^1} \sup_{k\in\mathbb{Z}^n} \|\varphi(\cdot - k)\, \widehat{f_\lambda}\|_{L^2} \\
&= C_n \lambda^{-n/2} \sup_{k\in\mathbb{Z}^n} \|\varphi(\lambda \cdot - k)\, \hat{f}\|_{L^2} \\
&= C_n \lambda^{-n/2} \sup_{k\in\mathbb{Z}^n} \left\| \varphi(\lambda \cdot - k) \left(\sum_{\ell\in\mathbb{Z}^n} \varphi(\cdot - \ell) \right) \hat{f} \right\|_{L^2}.
\end{aligned}$$

Since

$$\begin{aligned}
\left| \varphi(\lambda t - k) \left(\sum_{\ell\in\mathbb{Z}^n} \varphi(t - \ell) \right) \hat{f}(t) \right|^2 &\le 4^n \sum_{\ell\in\mathbb{Z}^n} \left| \varphi(\lambda t - k)\, \varphi(t-\ell)\, \hat{f}(t) \right|^2 \\
&= 4^n \sum_{\substack{|\ell_j - k_j/\lambda| \le 2/\lambda, \\ j=1,\cdots,n}} \left| \varphi(\lambda t - k)\, \varphi(t-\ell)\, \hat{f}(t) \right|^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\left\| \varphi(\lambda \cdot - k) \left(\sum_{\ell\in\mathbb{Z}^n} \varphi(\cdot - \ell) \right) \hat{f} \right\|_{L^2} \\
&\le \left(4^n \sum_{\substack{|\ell_j - k_j/\lambda| \le 2/\lambda, \\ j=1,\cdots,n}} \int_{\mathbb{R}^n} \left| \varphi(\lambda t - k)\, \varphi(t-\ell)\, \hat{f}(t) \right|^2 dt \right)^{1/2} \\
&\le \left(4^n (2\pi)^n \|\varphi\|_{L^\infty}^2 \sum_{\substack{|\ell_j - k_j/\lambda| \le 2/\lambda, \\ j=1,\cdots,n}} \|(M_\ell\Phi) * f\|_{L^2}^2 \right)^{1/2} \\
&\le \left(4^n (2\pi)^n \|\varphi\|_{L^\infty}^2 \left(\sup_{m\in\mathbb{Z}^n} \|(M_m\Phi) * f\|_{L^2} \right)^2 \sum_{\substack{|\ell_j - k_j/\lambda| \le 2/\lambda, \\ j=1,\cdots,n}} 1 \right)^{1/2} \\
&\le \left(C_n \|\varphi\|_{L^\infty}^2 \lambda^{-n} \left(\sup_{m\in\mathbb{Z}^n} \|(M_m\Phi) * f\|_{L^2} \right)^2 \right)^{1/2} \\
&= C_n \|\varphi\|_{L^\infty} \lambda^{-n/2} \sup_{m\in\mathbb{Z}^n} \|(M_m\Phi) * f\|_{L^2}.
\end{aligned}$$

Hence, by Lemma 3.3, we get

$$\|f_\lambda\|_{M^{2,\infty}} \le C_n \lambda^{-n} \sup_{m\in\mathbb{Z}^n} \|(M_m\Phi) * f\|_{L^2} \le C_n \lambda^{-n} \|f\|_{M^{2,\infty}}.$$

The proof is complete. □

Lemma 3.5. *Let $1 \le p \le \infty$. Then the following are true*:

(1) *If $p \le 2$, then there exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{p,1}} \le C\|f\|_{M^{p,1}} \quad \textit{for all } f \in M^{p,1}(\mathbb{R}^n) \textit{ and } \lambda \ge 1.$$

(2) *If $p \ge 2$, then there exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{p,1}} \le C\lambda^{-n(2/p-1)}\|f\|_{M^{p,1}} \quad \textit{for all } f \in M^{p,1}(\mathbb{R}^n) \textit{ and } \lambda \ge 1.$$

Proof. We first consider the case $p \le 2$. By Lemmas 2.2 and 2.4, and Lemma 3.4, we have

$$\begin{aligned}\|f_\lambda\|_{M^{2,1}} &= \sup |\langle f_\lambda, g\rangle_M| = \sup |\langle f_\lambda, g\rangle| \\ &= \lambda^{-n} \sup \left|\langle f, g_{1/\lambda}\rangle\right| \le \lambda^{-n} \sup \|f\|_{M^{2,1}} \|g_{1/\lambda}\|_{M^{2,\infty}} \\ &\le \lambda^{-n} \sup \|f\|_{M^{2,1}} \left(C\|g\|_{M^{2,\infty}}(1/\lambda)^{-n}\right) = C\|f\|_{M^{2,1}}\end{aligned}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \ge 1$, where the supremum is taken over all $g \in M^{2,\infty}(\mathbb{R}^n)$ such that $\|g\|_{M^{2,\infty}}(\mathbb{R}^n) = 1$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{2,1}(\mathbb{R}^n)$, this gives

$$\|f_\lambda\|_{M^{2,1}} \le C\|f\|_{M^{2,1}} \quad \text{for all } f \in M^{2,1}(\mathbb{R}^n) \text{ and } \lambda \ge 1. \tag{3.9}$$

On the other hand, by Lemma 3.1, we see that

$$\|f_\lambda\|_{M^{1,1}} \le C\|f\|_{M^{1,1}} \quad \text{for all } f \in M^{1,1}(\mathbb{R}^n) \text{ and } \lambda \ge 1. \tag{3.10}$$

Hence, by the interpolation theorem, (3.9) and (3.10) give Lemma 3.5 (1).

We next consider the case $p \ge 2$. By Lemma 3.1, we have

$$\|f_\lambda\|_{M^{\infty,1}} \le C\lambda^n\|f\|_{M^{\infty,1}} \quad \text{for all } f \in M^{\infty,1}(\mathbb{R}^n) \text{ and } \lambda \ge 1. \tag{3.11}$$

Therefore, by the interpolation theorem, (3.9) and (3.11) give

$$\|f_\lambda\|_{M^{p,1}} \le C \left(\lambda^0\right)^{1-\theta} (\lambda^n)^\theta \|f\|_{M^{p,1}}$$

for all $f \in M^{p,1}(\mathbb{R}^n)$ and $\lambda \ge 1$, where $1/p = (1-\theta)/2 + \theta/\infty$ and $0 \le \theta \le 1$. Since $\theta = -2/p + 1$, this implies Lemma 3.5 (2). The proof is complete. □

Lemma 3.6. *Let $1 \le p \le \infty$. Then the following are true*:

(1) *If $p \le 2$, then there exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{p,\infty}} \le C\lambda^{-2n/p}\|f\|_{M^{p,\infty}} \quad \textit{for all } f \in M^{p,\infty}(\mathbb{R}^n) \textit{ and } 0 < \lambda \le 1.$$

(2) *If $p \ge 2$, then there exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{p,\infty}} \le C\lambda^{-n}\|f\|_{M^{p,\infty}} \quad \textit{for all } f \in M^{p,\infty}(\mathbb{R}^n) \textit{ and } 0 < \lambda \le 1.$$

Proof. Let $1 < p \le 2$. By duality and Lemma 3.5 (2), we have

$$\begin{aligned}\|f_\lambda\|_{M^{p,\infty}} &= \sup |\langle f_\lambda, g\rangle| = \lambda^{-n} \sup \left|\langle f, g_{1/\lambda}\rangle\right| \\ &\le \lambda^{-n} \sup \|f\|_{M^{p,\infty}} \left(C(1/\lambda)^{-n(2/p'-1)}\|g\|_{M^{p',1}}\right) = C\lambda^{-2n/p}\|f\|_{M^{p,\infty}}\end{aligned}$$

for all $f \in M^{p,\infty}(\mathbb{R}^n)$ and $0 < \lambda \le 1$, where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p',1}} = 1$. In the case $p = 1$, by Lemma 3.1, we have

$$\|f_\lambda\|_{M^{1,\infty}} \le C\lambda^{-2n}\|f\|_{M^{1,\infty}} \quad \text{for all } f \in M^{1,\infty}(\mathbb{R}^n) \text{ and } 0 < \lambda \le 1.$$

Hence, we obtain Lemma 3.6 (1). In the same way, we can prove Lemma 3.6 (2). □

Lemma 3.7. *Let $1 \le p,q \le \infty$, $(p,q) \neq (1,\infty),(\infty,1)$ and $\epsilon>0$. Set*

$$f(t)=\sum_{k\neq 0}|k|^{-n/q-\epsilon}\,e^{ik\cdot t}\,\varphi(t) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

where φ is the Gauss function. Then $f \in M^{p,q}(\mathbb{R}^n)$ and there exists a constant $C>0$ such that $\|f_\lambda\|_{M^{p,q}} \ge C\lambda^{n(1/q-1)+\epsilon}$ for all $0<\lambda\le 1$.

Proof. We first prove $f \in M^{p,q}(\mathbb{R}^n)$. Since

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} e^{ik\cdot t}\,\varphi(t)\,\varphi(t-x)\,e^{-i\xi\cdot t}\,dt\right| \\
&=\left|\int_{\mathbb{R}^n}\varphi(t)\,\varphi(x-t)\left\{(1+|\xi-k|^2)^{-n}(I-\Delta_t)^n\,e^{-i(\xi-k)\cdot t}\right\}dt\right| \\
&=(1+|\xi-k|^2)^{-n}\left|\sum_{|\beta_1+\beta_2|\le 2n} C_{\beta_1,\beta_2}\int_{\mathbb{R}^n}(\partial^{\beta_1}\varphi)(t)\,(\partial^{\beta_2}\varphi)(x-t)\,e^{-i(\xi-k)\cdot t}\,dt\right| \\
&\le C(1+|\xi-k|^2)^{-n}\sum_{|\beta_1+\beta_2|\le 2n}|\partial^{\beta_1}\varphi|*|\partial^{\beta_2}\varphi|(x),
\end{aligned}$$

we have

$$\begin{aligned}
&\|f\|_{M^{p,q}}=\|V_\varphi f\|_{L^{p,q}} \\
&=\left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|\sum_{k\neq 0}|k|^{-n/q-\epsilon}\int_{\mathbb{R}^n}e^{ik\cdot t}\,\varphi(t)\,\varphi(t-x)\,e^{-i\xi\cdot t}\,dt\right|dx\right)^{q/p}d\xi\right\}^{1/q} \\
&\le C\left\{\int_{\mathbb{R}^n}\left(\sum_{k\neq 0}|k|^{-n/q-\epsilon}\,(1+|\xi-k|^2)^{-n}\right)^q d\xi\right\}^{1/q} \\
&=C\left\{\sum_{\ell\in\mathbb{Z}^n}\int_{\ell+[-1/2,1/2]^n}\left(\sum_{k\neq 0}|k|^{-n/q-\epsilon}\,(1+|\xi-k|^2)^{-n}\right)^q d\xi\right\}^{1/q} \\
&\le C\left\{\sum_{\ell\in\mathbb{Z}^n}\left(\sum_{k\neq 0}|k|^{-n/q-\epsilon}\,(1+|\ell-k|^2)^{-n}\right)^q\right\}^{1/q}.
\end{aligned}$$

Since $\{|k|^{-n/q-\epsilon}\}_{k\neq 0} \in \ell^q(\mathbb{Z}^n)$, by Young's inequality, we see that $f \in M^{p,q}(\mathbb{R}^n)$.

We next consider the second part. Since $\varphi \in M^{p',q'}(\mathbb{R}^n)$, by duality, we have

$$\begin{aligned}
\|f_\lambda\|_{M^{p,q}} &= \sup_{\|g\|_{M^{p',q'}}=1}|\langle f_\lambda, g\rangle| \ge |\langle f_\lambda,\varphi\rangle| \\
&=\left|\sum_{k\neq 0}|k|^{-n/q-\epsilon}\int_{\mathbb{R}^n}e^{i(\lambda k)\cdot t}\,\varphi(\lambda t)\,\varphi(t)\,dt\right| \\
&=\pi^{n/2}(1+\lambda^2)^{-n/2}\sum_{k\neq 0}|k|^{-n/q-\epsilon}\,e^{-\frac{\lambda^2|k|^2}{4(1+\lambda^2)}} \\
&\ge (\pi/2)^{n/2}\sum_{\substack{0<|k_j|\le 1/\lambda,\\ j=1,\cdots,n}}|k|^{-n/q-\epsilon}\,e^{-\frac{\lambda^2|k|^2}{4(1+\lambda^2)}}
\end{aligned}$$

$$\geq C\lambda^{n/q+\epsilon} \sum_{\substack{0<|k_j|\leq 1/\lambda, \\ j=1,\cdots,n}} 1 \geq C\lambda^{n(1/q-1)+\epsilon}$$

for all $0 < \lambda \leq 1$. The proof is complete. □

We are now ready to prove Theorem 1.1 (1) with $(1/p, 1/q) \in I_2^*$ and (2) with $(1/p, 1/q) \in I_2$.

Proof of Theorem 1.1 (2) with $(1/p, 1/q) \in I_2$. We note that $\mu_2(p, q) = 1/q - 1$ if $(1/p, 1/q) \in I_2$. Let $p \geq 2$ and $1 \leq q \leq \infty$ be such that $(1/p, 1/q) \in I_2$ and $1/p \leq 1/q$. If $q = 1$ then $p = \infty$, and we have already proved this case in Theorem 1.1 (2) with $(1/p, 1/q) \in I_1$. Hence, we may assume $1 < q \leq \infty$. We note that $1 \leq p' \leq 2$ and $1 \leq q' < \infty$. Since $(1/p', 1/q') \in I_2^*$ and $1/p' \geq 1/q'$, by duality and Lemma 3.2, we have

$$\begin{aligned}
\|f_\lambda\|_{M^{p,q}} &= \sup |\langle f_\lambda, g\rangle| = \lambda^{-n} \sup \left|\langle f, g_{1/\lambda}\rangle\right| \\
&\leq \lambda^{-n} \sup \|f\|_{M^{p,q}} \|g_{1/\lambda}\|_{M^{p',q'}} \\
&\leq \lambda^{-n} \sup \|f\|_{M^{p,q}} \left(C(\lambda^{-1})^{-n(2/p'-1/q')}(1+\lambda^{-2})^{n(1/p'-1/2)}\|g\|_{M^{p',q'}}\right) \\
&\leq C\lambda^{n(1/q-1)}\|f\|_{M^{p,q}}
\end{aligned} \tag{3.12}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \leq 1$, where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p',q'}} = 1$. This is the first part of Theorem 1.1 (2) with $(1/p, 1/q) \in I_2$ and $1/p \leq 1/q$. Let $p \geq 2$ and $2 \leq q < \infty$ be such that $(1/p, 1/q) \in I_2$ and $1/p \geq 1/q$. From (3.12) it follows that

$$\|f_\lambda\|_{M^{r,r}} \leq C\lambda^{n(1/r-1)}\|f\|_{M^{r,r}} \quad \text{for all } f \in M^{r,r}(\mathbb{R}^n) \text{ and } 0 < \lambda \leq 1, \tag{3.13}$$

where $2 \leq r \leq \infty$. Take $2 \leq r \leq \infty$ and $0 \leq \theta \leq 1$ such that $1/p = (1-\theta)/2 + \theta/r$ and $1/q = (1-\theta)/\infty + \theta/r$. Since $q \neq \infty$, we note that $r \neq \infty$. Then, by the interpolation theorem, Lemma 3.4 and (3.13) give

$$\|f_\lambda\|_{M^{p,q}} \leq C\left(\lambda^{-n}\right)^{1-\theta}\left(\lambda^{n(1/r-1)}\right)^{\theta}\|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \leq 1$. Since $\theta/r = 1/q$, we have

$$\|f_\lambda\|_{M^{p,q}} \leq C\lambda^{n(\theta/r-1)}\|f\|_{M^{p,q}} \leq C\lambda^{n(1/q-1)}\|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \leq 1$. In the case $q = \infty$, by Lemma 3.6 (2), we have nothing to prove. Hence, we obtain the first part of Theorem 1.1 (2) with $(1/p, 1/q) \in I_2$ and $1/p \geq 1/q$.

We next consider the second part of Theorem 1.1 (2) with $(1/p, 1/q) \in I_2$. Let $p \geq 2$ and $1 \leq q \leq \infty$ be such that $(1/p, 1/q) \in I_2$. Since $(1/\infty, 1/1) \in I_1$, we may assume $(p, q) \neq (\infty, 1)$. Assume that there exist constants $C > 0$ and $\beta \in \mathbb{R}$ such that

$$\|f_\lambda\|_{M^{p,q}} \leq C\lambda^{\beta}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \leq 1,$$

where $\beta > n(1/q-1)$. Then we can take $\epsilon > 0$ such that $n(1/q-1) + \epsilon < \beta$. For this ϵ, we set

$$f(t) = \sum_{k\neq 0} |k|^{-n/q-\epsilon} e^{ik\cdot t} \varphi(t),$$

where φ is the Gauss function. Then, by Lemma 3.7, we see that $f \in M^{p,q}(\mathbb{R}^n)$ and there exists a constant $C' > 0$ such that $\|f_\lambda\|_{M^{p,q}} \geq C'\lambda^{n(1/q-1)+\epsilon}$ for all $0 < \lambda \leq 1$. Hence,

$$C'\lambda^{n(1/q-1)+\epsilon} \leq \|f_\lambda\|_{M^{p,q}} \leq C\lambda^{\beta}\|f\|_{M^{p,q}}$$

for all $0<\lambda\le 1$. However, since $n(1/q-1)+\epsilon<\beta$, this is contradiction. Therefore, β must satisfy $\beta\le n(1/q-1)$. The proof is complete.

Proof of Theorem 1.1 (1) with $(1/p,1/q)\in I_2^*$. We note that $\mu_1(p,q)=1/q-1$ if $(1/p,1/q)\in I_2^*$. In every case except for $(p,q)\neq(1,\infty)$, by duality, Theorem 1.1 (2) with $(1/p',1/q')\in I_2$ and the same argument as in the proof of Theorem 1.1 (1) with $(1/p,1/q)\in I_1^*$, we can prove Theorem 1.1 (1) with $(1/p,1/q)\in I_2^*$. For the case $(p,q)=(1,\infty)$, we have already proved in Theorem 1.1 (1) with $(1/p,1/q)\in I_1^*$.

Our last goal of this section is to prove Theorem 1.1 (1) with $(1/p,1/q)\in I_3^*$ and (2) with $(1/p,1/q)\in I_3$. In the following lemma, we use the fact that there exists $\varphi\in\mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi\subset[-1/8,1/8]^n$ and $|\hat{\varphi}|\ge 1$ on $[-2,2]^n$ (see, for example, the proof of [3, Theorem 2.6]).

Lemma 3.8. *Let* $1\le p\le\infty$, $1\le q<\infty$ *and* $\epsilon>0$. *Suppose that* $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ *satisfy* $\operatorname{supp}\varphi\subset[-1/8,1/8]^n$, $\operatorname{supp}\psi\subset[-1/2,1/2]^n$, $|\hat{\varphi}|\ge 1$ *on* $[-2,2]^n$ *and* $\psi=1$ *on* $[-1/4,1/4]^n$. *Set*

$$f(t)=\sum_{k\neq 0}|k|^{-n/q-\epsilon}\,e^{ik\cdot t}\,\psi(t-k)\quad in\quad \mathcal{S}'(\mathbb{R}^n).$$

Then $f\in M^{p,q}(\mathbb{R}^n)$ *and there exists a constant* $C>0$ *such that*

$$\|V_\varphi(f_\lambda)\|_{L^{p,q}}\ge C\lambda^{-n(2/p-1/q)+\epsilon}\quad for\ all\ 0<\lambda\le 1.$$

Proof. In the same way as in the proof of Lemma 3.7, we can prove $f\in M^{p,q}(\mathbb{R}^n)$. We consider the second part. Since $\|V_\varphi(f_\lambda)\|_{L^{p,q}}=\lambda^{-n(1/p-1/q+1)}\|V_{\varphi_{1/\lambda}}f\|_{L^{p,q}}$, it is enough to show that $\|V_{\varphi_{1/\lambda}}f\|_{L^{p,q}}\ge C\lambda^{-n/p+n+\epsilon}$ for all $0<\lambda\le 1$. We note that $\operatorname{supp}\varphi((\cdot-x)/\lambda)\subset\ell+[-1/4,1/4]^n$ for all $0<\lambda\le 1,\ell\in\mathbb{Z}^n$ and $x\in\ell+[-1/8,1,8]^n$. Since $\operatorname{supp}\psi(\cdot-k)\subset k+[-1/2,1/2]^n$ and $\psi(t-k)=1$ if $t\in k+[-1/4,1/4]^n$, it follows that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\left|V_{\varphi_{1/\lambda}}f(x,\xi)\right|^p dx\right)^{1/p}\\
&=\left(\int_{\mathbb{R}^n}\left|\sum_{k\neq 0}|k|^{-n/q-\epsilon}\int_{\mathbb{R}^n}e^{ik\cdot t}\,\psi(t-k)\,\overline{\varphi\left(\frac{t-x}{\lambda}\right)}\,e^{-i\xi\cdot t}\,dt\right|^p dx\right)^{1/p}\\
&\ge\left(\sum_{\ell\neq 0}\int_{\ell+[-1/8,1/8]^n}\left|\sum_{k\neq 0}|k|^{-n/q-\epsilon}\int_{\mathbb{R}^n}e^{-i(\xi-k)\cdot t}\,\psi(t-k)\,\overline{\varphi\left(\frac{t-x}{\lambda}\right)}\,dt\right|^p dx\right)^{1/p}\\
&=\left(\sum_{\ell\neq 0}\int_{\ell+[-1/8,1/8]^n}\left||\ell|^{-n/q-\epsilon}\int_{\mathbb{R}^n}e^{-i(\xi-\ell)\cdot t}\,\overline{\varphi\left(\frac{t-x}{\lambda}\right)}\,dt\right|^p dx\right)^{1/p}\\
&=4^{-n/p}\left(\sum_{\ell\neq 0}\left||\ell|^{-n/q-\epsilon}\lambda^n\hat{\varphi}(-\lambda(\xi-\ell))\right|^p\right)^{1/p}.
\end{aligned}$$

Hence, using $|\hat{\varphi}|\ge 1$ on $[-2,2]^n$, we get

$$\|V_{\varphi_{1/\lambda}}f\|_{L^{p,q}}\ge 4^{-n/p}\lambda^n\left\{\int_{\mathbb{R}^n}\left(\sum_{\ell\neq 0}\left||\ell|^{-n/q-\epsilon}\hat{\varphi}(-\lambda(\xi-\ell))\right|^p\right)^{q/p}d\xi\right\}^{1/q}$$

$$
\begin{aligned}
&= 4^{-n/p}\lambda^{n-n/q}\left\{\int_{\mathbb{R}^n}\left(\sum_{\ell\neq 0}\left||\ell|^{-n/q-\epsilon}\hat{\varphi}(\xi+\lambda\ell))\right|^p\right)^{q/p} d\xi\right\}^{1/q} \\
&\geq 4^{-n/p}\lambda^{n-n/q}\left\{\int_{[-1,1]^n}\left(\sum_{\substack{0<|\ell_j|\leq 1/\lambda,\\ j=1,\cdots,n}}\left||\ell|^{-n/q-\epsilon}\hat{\varphi}(\xi+\lambda\ell))\right|^p\right)^{q/p} d\xi\right\}^{1/q} \\
&\geq 4^{-n/p}2^{n/q}\lambda^{n-n/q}\left(\sum_{\substack{0<|\ell_j|\leq 1/\lambda,\\ j=1,\cdots,n}}|\ell|^{-(n/q+\epsilon)p}\right)^{1/p} \\
&\geq C_n\lambda^{n-n/q}\lambda^{n/q+\epsilon}\left(\sum_{\substack{0<|\ell_j|\leq 1/\lambda,\\ j=1,\cdots,n}}1\right)^{1/p} \geq C_n\lambda^{-n/p+n+\epsilon}
\end{aligned}
$$

for all $0<\lambda\leq 1$. The proof is complete. □

For Lemma 3.8, we do not need $\epsilon>0$ in the case $q=\infty$.

Lemma 3.9. *Let $1\leq p\leq\infty$. Suppose that $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ are as in Lemma 3.8. Set*

$$f(t)=\sum_{k\in\mathbb{Z}^n}e^{ik\cdot t}\,\psi(t-k)\quad \textit{in}\quad \mathcal{S}'(\mathbb{R}^n).$$

Then $f\in M^{p,\infty}(\mathbb{R}^n)$ and there exists a constant $C>0$ such that $\|f_\lambda\|_{M^{p,\infty}}\geq C\lambda^{-2n/p}$ for all $0<\lambda\leq 1$. In particular, if $1\leq p\leq 2$ then there exist constants $C,C'>0$ such that

$$C\lambda^{-2n/p}\leq\|f_\lambda\|_{M^{p,\infty}}\leq C'\lambda^{-2n/p}\quad \textit{for all } 0<\lambda\leq 1.$$

Proof. In the same way as in the proof of Lemma 3.7, we can prove

$$\left|\int_{\mathbb{R}^n}e^{ik\cdot t}\,\psi(t-k)\,\overline{\varphi(t-x)}\,e^{-i\xi\cdot t}\,dt\right|\leq C(1+|x-k|^2)^{-n}\,(1+|\xi-k|^2)^{-n}.$$

Hence,

$$
\begin{aligned}
|V_\varphi f(x,\xi)| &= \left|\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n}e^{ik\cdot t}\,\psi(t-k)\,\overline{\varphi(t-x)}\,e^{-i\xi\cdot t}\,dt\right| \\
&\leq C\sum_{k\in\mathbb{Z}^n}(1+|x-k|^2)^{-n}\,(1+|\xi-k|^2)^{-n}\leq C(1+|x-\xi|^2)^{-n}
\end{aligned}
$$

for all $x,\xi\in\mathbb{R}^n$. This implies $f\in M^{p,\infty}(\mathbb{R}^n)$.

We next consider the second part. Since $\|V_{\varphi_{1/\lambda}}f(\cdot,\xi)\|_{L^p}$ is continuous with respect to $\xi\in\mathbb{R}^n$, we see that $\|V_{\varphi_{1/\lambda}}f\|_{L^{p,\infty}}=\sup_{\xi\in\mathbb{R}^n}\|V_{\varphi_{1/\lambda}}f(\cdot,\xi)\|_{L^p}$ for each $0<\lambda\leq 1$. Hence, by the same argument as in the proof of Lemma 3.8, we have

$$\|V_\varphi(f_\lambda)\|_{L^{p,\infty}}=\lambda^{-n(1/p+1)}\|V_{\varphi_{1/\lambda}}f\|_{L^{p,\infty}}\geq\lambda^{-n(1/p+1)}\|V_{\varphi_{1/\lambda}}f(\cdot,0)\|_{L^p}$$

$$\geq C\lambda^{-n(1/p+1)} \left(\sum_{\ell \in \mathbb{Z}^n} |\lambda^n \hat{\varphi}(\lambda \ell)|^p \right)^{1/p} \geq C\lambda^{-n/p} \left(\sum_{\substack{|\ell_j| \leq 1/\lambda, \\ j=1,\cdots,n}} |\hat{\varphi}(\lambda \ell)|^p \right)^{1/p} \geq C\lambda^{-2n/p}$$

for all $0 < \lambda \leq 1$. Combining Lemma 3.6 (1), we get $\|f\|_{M^{p,\infty}} \sim \lambda^{-2n/p}$ in the case $0 < \lambda \leq 1$. The proof is complete. $\square$

We are now ready to prove Theorem 1.1 (1) with $(1/p, 1/q) \in I_3^*$ and (2) with $(1/p, 1/q) \in I_3$.

Proof of Theorem 1.1 (2) with $(1/p, 1/q) \in I_3$. We note that $\mu_2(p, q) = -2/p + 1/q$ if $(1/p, 1/q) \in I_3$. Let $1 \leq p \leq 2$ and $1 \leq q \leq \infty$ be such that $(1/p, 1/q) \in I_3$ and $1/p + 1/q \geq 1$. We note that, if $(1/p, 1/q) \in I_3$ and $1/p + 1/q \geq 1$, then $(1/p, 1/q) \in I_2^*$ and $1/p \geq 1/q$. Then, by Lemma 3.2, there exists a constant $C > 0$ such that

$$\|f_\lambda\|_{M^{p,q}} \leq C\lambda^{-n(2/p-1/q)} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \leq 1. \tag{3.14}$$

This is the first part of Theorem 1.1 (2) with $(1/p, 1/q) \in I_3$ and $1/p + 1/q \geq 1$. Let $1 \leq p \leq 2$ and $2 \leq q < \infty$ be such that $(1/p, 1/q) \in I_3$ and $1/p + 1/q \leq 1$. (3.14) implies

$$\|f_\lambda\|_{M^{r,r'}} \leq C\lambda^{-n(2/r-1/r')} \|f\|_{M^{r,r'}} = C\lambda^{-n(3/r-1)} \|f\|_{M^{r,r'}} \tag{3.15}$$

for all $f \in M^{r,r'}(\mathbb{R}^n)$ and $0 < \lambda \leq 1$, where $1 \leq r \leq 2$. Take $1 \leq r \leq 2$ and $0 \leq \theta \leq 1$ such that $1/p = (1-\theta)/2 + \theta/r$ $1/q = (1-\theta)/\infty + \theta/r'$. Since $q \neq \infty$, we note that $r' \neq \infty$. Then, by the interpolation theorem, Lemma 3.4 and (3.15) give

$$\|f_\lambda\|_{M^{p,q}} \leq C \left(\lambda^{-n}\right)^{1-\theta} \left(\lambda^{-n(3/r-1)}\right)^{\theta} \|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \leq 1$. Since $1 - \theta = 2(1/p - \theta/r)$ and $\theta/r = \theta - 1/q$, we have

$$\begin{aligned} \|f_\lambda\|_{M^{p,q}} &\leq C\lambda^{-n(2(1/p-\theta/r)+3\theta/r-\theta)} \|f\|_{M^{p,q}} \\ &= C\lambda^{-n(2/p+\theta/r-\theta)} \|f\|_{M^{p,q}} = C\lambda^{-n(2/p-1/q)} \|f\|_{M^{p,q}} \end{aligned}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \leq 1$. In the case $q = \infty$, by Lemma 3.6 (1), we have nothing to prove. Hence, we obtain the first part of Theorem 1.1 (2) with $(1/p, 1/q) \in I_3$ and $1/p + 1/q \leq 1$.

By using Lemma 3.8 (or 3.9), we can prove the second part of Theorem 1.1 (2) with $(1/p, 1/q) \in I_3$ in the same way as in the proof of the second part of Theorem 1.1 (2) with $(1/p, 1/q) \in I_2$. We omit the proof.

Proof of Theorem 1.1 (1) with $(1/p, 1/q) \in I_3^*$. We note that $\mu_1(p, q) = -2/p + 1/q$ if $(1/p, 1/q) \in I_3^*$. In every case except for $(p, q) \neq (\infty, 1)$, by duality, Theorem 1.1 (2) with $(1/p', 1/q') \in I_3$ and the same argument as in the proof of Theorem 1.1 (1) with $(1/p, 1/q) \in I_1^*$, we can prove Theorem 1.1 (1) with $(1/p, 1/q) \in I_3^*$.

For the first part of Theorem 1.1 (1) with $(p, q) = (\infty, 1)$, by (3.11), we have nothing to prove. By using the interpolation theorem, we can prove the second part in the same way as in the proof of Theorem 1.1 (1) with $q = \infty$.

4. The inclusion between Besov spaces and modulation spaces

In this section, we prove Theorem 1.2 which appeared in the introduction. It is sufficient to prove the first statement only because the first one implies the second one by the duality argument and the elementary relation

$$\nu_2(p,q) = -\nu_1(p',q').$$

See also Section 2 for the dual spaces of the modulation spaces (Lemma 2.4) and Besov spaces.

For the preparation to prove Theorem 1.2 (1) with $(1/p,1/q) \in I_1^*$, we show three lemmas in the below. We denote by B the tensor product of B-spline of degree 2, that is

$$B(t) = \prod_{j=1}^{n} \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}(t_j),$$

where $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$. We note that $\operatorname{supp} B \subset [-1,1]^n$ and $\mathcal{F}^{-1}B \in M^{p,q}(\mathbb{R}^n)$ for all $1 \le p,q \le \infty$.

Lemma 4.1. *Let $1 \le p,q \le \infty$, $(p,q) \neq (1,\infty),(\infty,1)$ and $\epsilon > 0$. Suppose that $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\psi = 1$ on $\{\xi : |\xi| \le 1/2\}$ and $\operatorname{supp}\psi \subset \{\xi : |\xi| \le 1\}$. Set*

$$f(t) = \sum_{\ell \neq 0} |\ell|^{-n/p-\epsilon}\, \Psi(t-\ell) \quad \textit{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

where $\Psi = \mathcal{F}^{-1}\psi$. Then $f \in M^{p,q}(\mathbb{R}^n)$ and there exists a constant $C > 0$ such that $\|f_\lambda\|_{M^{p,q}} \ge C\lambda^{-n/p-\epsilon}$ for all $\lambda \ge 2\sqrt{n}$.

Proof. In the same way as in the proof of Lemma 3.7, we can prove $f \in M^{p,q}(\mathbb{R}^n)$. We consider the second part. Let $\lambda \ge 2\sqrt{n}$. Since $\psi(\cdot/\lambda) = 1$ on $[-1,1]^n$, we have

$$\int_{\mathbb{R}^n} \Psi(\lambda t - \ell)\,(\mathcal{F}^{-1}B)(t)\,dt = (2\pi)^{-n}\lambda^{-n} \int_{\mathbb{R}^n} e^{-i(\ell/\lambda)\cdot t}\,\psi(t/\lambda)\,B(t)\,dt$$
$$= (2\pi)^{-n}\lambda^{-n} \int_{\mathbb{R}^n} e^{-i(\ell/\lambda)\cdot t}\,B(t)\,dt = (2\pi)^{-n}\lambda^{-n} \prod_{j=1}^{n} \left(\frac{\sin \ell_j/2\lambda}{\ell_j/2\lambda}\right)^2.$$

We note that $\prod_{j=1}^n \{(\sin\xi_j)/\xi_j\}^2 \ge C$ on $[-\pi/2,\pi/2]^n$ for some constant $C > 0$. Since $\mathcal{F}^{-1}B \in M^{p',q'}(\mathbb{R}^n)$, by Lemmas 2.2 and 2.4, we get

$$\|f_\lambda\|_{M^{p,q}} = \sup_{\|g\|_{M^{p',q'}}=1} |\langle f_\lambda, g\rangle_M| \ge \|\mathcal{F}^{-1}B\|_{M^{p',q'}}^{-1} \left|\langle f_\lambda, \mathcal{F}^{-1}B\rangle_M\right|$$
$$= \|\mathcal{F}^{-1}B\|_{M^{p',q'}}^{-1} \left| \sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon} \frac{1}{\|\Phi\|_{L^2}^2} \int_{\mathbb{R}^{2n}} V_\Phi[\Psi(\lambda\cdot-\ell)](x,\xi)\,\overline{V_\Phi[\mathcal{F}^{-1}B](x,\xi)}\,dx\,d\xi \right|$$
$$= \|\mathcal{F}^{-1}B\|_{M^{p',q'}}^{-1} \left| \sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon} \int_{\mathbb{R}^n} \Psi(\lambda t-\ell)\,(\mathcal{F}^{-1}B)(t)\,dt \right|$$
$$= (2\pi)^{-n}\|\mathcal{F}^{-1}B\|_{M^{p',q'}}^{-1}\lambda^{-n} \left| \sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon} \prod_{j=1}^{n}\left(\frac{\sin \ell_j/2\lambda}{\ell_j/2\lambda}\right)^2 \right|$$
$$\ge (2\pi)^{-n}\|\mathcal{F}^{-1}B\|_{M^{p',q'}}^{-1}\lambda^{-n} \sum_{\substack{0<|\ell_j|\le\lambda\pi,\\ j=1,\cdots,n}} |\ell|^{-n/p-\epsilon} \prod_{j=1}^{n}\left(\frac{\sin \ell_j/2\lambda}{\ell_j/2\lambda}\right)^2$$

$$\geq C\lambda^{-n}\lambda^{-n/p-\epsilon} \sum_{\substack{0<|\ell_j|\leq\lambda\pi,\\ j=1,\cdots,n}} 1 \geq C\lambda^{-n/p-\epsilon}$$

for all $\lambda \geq 2\sqrt{n}$. The proof is complete. $\square$

Lemma 4.2. *Suppose that $1 \leq p,q \leq \infty$, $(p,q) \neq (1,\infty),(\infty,1)$ and $\epsilon > 0$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in Lemma 4.1. Set*

$$f(t) = e^{8it_1} \sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon}\, \Psi(t-\ell) \quad in \quad \mathcal{S}'(\mathbb{R}^n),$$

where $t=(t_1,\cdots,t_n)\in\mathbb{R}^n$ and $\Psi=\mathcal{F}^{-1}\psi$. Then $f\in M^{p,q}(\mathbb{R}^n)$ and there exists a constant $C>0$ such that $\|f_\lambda\|_{M^{p,q}} \geq C\lambda^{-n/p-\epsilon}$ for all $\lambda \geq 2\sqrt{n}$.

Proof. Let $g(t)=\sum_{\ell\neq 0}|\ell|^{-n/p-\epsilon}\,\Psi(t-\ell)$. Since $f=M_{8e_1}g$ and $f_\lambda = M_{8\lambda e_1}g_\lambda$, we have $V_\Phi(f_\lambda)(x,\xi) = V_\Phi(g_\lambda)(x,\xi-8\lambda e_1)$, where $e_1=(1,0,\cdots,0)$. This gives $\|f_\lambda\|_{M^{p,q}} = \|g_\lambda\|_{M^{p,q}}$. Hence, by Lemma 4.1, we obtain Lemma 4.2. $\square$

Lemma 4.3. *Suppose that $1\leq p,q\leq\infty$, $s\in\mathbb{R}$ and $\epsilon>0$. Let $\psi\in\mathcal{S}(\mathbb{R}^n)$ be as in Lemma 4.1. Set*

$$f(t) = e^{8it_1} \sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon}\, \Psi(t-\ell) \quad in \quad \mathcal{S}'(\mathbb{R}^n),$$

where $t=(t_1,\cdots,t_n)\in\mathbb{R}^n$ and $\Psi=\mathcal{F}^{-1}\psi$. Then there exists a constant $C>0$ such that $\|f_{2^k}\|_{B^{p,q}_s} \leq C2^{k(s-n/p)}$ for all $k\in\mathbb{Z}_+$.

Proof. Let $k\in\mathbb{Z}_+$. Since $\operatorname{supp}\varphi_0 \subset \{\xi : |\xi|\leq 2\}$, $\operatorname{supp}\varphi_j \subset \{\xi : 2^{j-1}\leq|\xi|\leq 2^{j+1}\}$ if $j\geq 1$, and $\operatorname{supp}\psi(\cdot/2^k - 8e_1) \subset \{\xi : |\xi - 2^{k+3}e_1| \leq 2^k\}$, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi_j(x-t)\, \left(e^{8i(2^k t_1)}\, \Psi(2^k t-\ell)\right) dt \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot t}\, \varphi_j(t)\, \left(2^{-kn} e^{-i\ell\cdot(t/2^k-8e_1)}\, \psi(t/2^k-8e_1)\right) dt \\
&= \begin{cases} (2\pi)^{-n} e^{8i\ell_1} \int_{\mathbb{R}^n} e^{i(2^k x-\ell)\cdot t}\, \varphi_j(2^k t)\, \psi(t-8e_1)\, dt, & \text{if} \quad k+2\leq j\leq k+4 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
&|\Phi_j * (f_{2^k})(x)| = \left| \sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon} \int_{\mathbb{R}^n} \Phi_j(x-t)\, \left(e^{8i(2^k t_1)}\, \Psi(2^k t-\ell)\right) dt \right| \\
&\leq C \sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon} \left| \int_{\mathbb{R}^n} \left\{ (1+|2^k x-\ell|^2)^{-n} (I-\Delta_t)^n e^{i(2^k x-\ell)\cdot t} \right\} \varphi_j(2^k t)\, \psi(t-8e_1)\, dt \right| \\
&\leq C \sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon}\, (1+|2^k x-\ell|^2)^{-n},
\end{aligned}$$

where $k+2\leq j\leq k+4$. On the other hand, $\Phi_j*(f_{2^k})=0$ if $j<k+2$ or $j>k+4$. Thus, $\|\Phi_j*(f_{2^k})\|_{L^p} \leq C2^{-kn/p}$ if $k+2\leq j\leq k+4$, and $\|\Phi_j*(f_{2^k})\|_{L^p}=0$ if $j<k+2$ or $j>k+4$. Therefore,

$$\|f_{2^k}\|_{B^{p,q}_s} = \left(\sum_{j=k+2}^{k+4} 2^{jsq} \|\Phi_j*(f_{2^k})\|_{L^p}^q \right)^{1/q} \leq C2^{-kn/p} \left(\sum_{j=k+2}^{k+4} 2^{jsq} \right)^{1/q} \leq C2^{k(s-n/p)}.$$

The proof is complete. $\square$

We are now ready to prove Theorem 1.2 (1) with $(1/p, 1/q) \in I_1^*$.

Proof of Theorem 1.2 (1) with $(1/p, 1/q) \in I_1^*$. Let $(1/p, 1/q) \in I_1^*$ and $(p, q) \neq (1, \infty)$. Then $\nu_1(p, q) = 0$. We assume that $B_s^{p,q}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n)$, where $s < 0$. Set $s = -\epsilon$, where $\epsilon > 0$. For this ϵ, we define f by

$$f(t) = e^{8it_1} \sum_{\ell \neq 0} |\ell|^{-n/p-\epsilon/2}\, \Psi(t-\ell),$$

where $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$, $\Psi = \mathcal{F}^{-1}\psi$ and ψ is as in Lemma 4.1. Then, by Lemmas 4.2 and 4.3, we have

$$C_1 2^{-k(n/p+\epsilon/2)} \le \|f_{2^k}\|_{M^{p,q}} \le C_2 \|f_{2^k}\|_{B_s^{p,q}} \le C_3 2^{k(s-n/p)} = C_3 2^{-k(n/p+\epsilon)}$$

for any large integer k. However, this is contradiction. Hence, s must satisfy $s \ge 0$.

We next consider the case $(p, q) = (1, \infty)$. Assume that $B_s^{1,\infty}(\mathbb{R}^n) \hookrightarrow M^{1,\infty}(\mathbb{R}^n)$. Let $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ be such that $\operatorname{supp} \psi \subset \{\xi : 1/2 \le |\xi| \le 2\}$. Since $M^{1,\infty}(\mathbb{R}^n) \hookrightarrow \mathcal{F}L^\infty(\mathbb{R}^n)$ ([9, Proposition 1.7]), we see that

$$2^{-kn}\|\psi\|_{L^\infty} = \|\mathcal{F}[\Psi_{2^k}]\|_{L^\infty} \le C\|\Psi_{2^k}\|_{M^{1,\infty}} \quad \text{for all } k \in \mathbb{Z}_+,$$

where $\Psi = \mathcal{F}^{-1}\psi$. On the other hand, it is easy to show that

$$\|\Psi_{2^k}\|_{B_s^{1,\infty}} \le C 2^{k(s-n)} \quad \text{for all } k \in \mathbb{Z}_+.$$

Hence, by our assumption, we get

$$2^{-kn}\|\psi\|_{L^\infty} \le C_1\|\Psi_{2^k}\|_{M^{1,\infty}} \le C_2\|\Psi_{2^k}\|_{B_s^{1,\infty}} \le C_3 2^{k(s-n)}$$

for all $k \in \mathbb{Z}_+$. This implies $s \ge 0$. The proof is complete.

Our next goal is to prove Theorem 1.2 (1) with $(1/p, 1/q) \in I_2^*$. We remark the following fact, and give the proof for reader's convenience.

Lemma 4.4 ([7, Proposition 1.1])**.** *Let* $1 \le p, q \le \infty$ *and* $s > 0$. *Then there exists a constant* $C > 0$ *such that*

$$\|f_\lambda\|_{B_s^{p,q}} \le C\lambda^{s-n/p}\|f\|_{B_s^{p,q}} \quad \textit{for all } f \in B_s^{p,q}(\mathbb{R}^n) \textit{ and } \lambda \ge 1.$$

Proof. Let $j_0 \in \mathbb{Z}_+$ be such that $2^{j_0} \le \lambda < 2^{j_0+1}$. Since $\sum_{j=0}^\infty \varphi_j(\xi) = 1$ for all $\xi \in \mathbb{R}^n$, we see that

$$\varphi_j(\lambda\xi) = \sum_{\ell=-2}^{1} \varphi_j(\lambda\xi)\, \varphi_{j+\ell}(2^{j_0}\xi) \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}_+,$$

where $\varphi_{j+\ell} = 0$ if $j + \ell < 0$. Hence, by Young's inequality, we have

$$\begin{aligned}
\|f_\lambda\|_{B_s^{p,q}} &= \left(\sum_{j=0}^\infty 2^{jsq}\|\mathcal{F}^{-1}[\varphi_j\, \widehat{f_\lambda}]\|_{L^p}^q\right)^{1/q} = \lambda^{-n/p}\left(\sum_{j=0}^\infty 2^{jsq}\|\mathcal{F}^{-1}[\varphi_j(\lambda\cdot)\, \hat{f}]\|_{L^p}^q\right)^{1/q} \\
&\le \lambda^{-n/p}\sum_{\ell=-2}^{1}\left(\sum_{j=0}^\infty 2^{jsq}\|\mathcal{F}^{-1}[\varphi_j(\lambda\cdot)\, \varphi_{j+\ell}(2^{j_0}\cdot)\, \hat{f}]\|_{L^p}^q\right)^{1/q} \\
&\le \lambda^{-n/p}\sum_{\ell=-2}^{1}\left\{\sum_{j=0}^\infty 2^{jsq}\left(\|\mathcal{F}^{-1}[\varphi_j(\lambda\cdot)]\|_{L^1}\|\mathcal{F}^{-1}[\varphi_{j+\ell}(2^{j_0}\cdot)\, \hat{f}]\|_{L^p}\right)^q\right\}^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\le C\lambda^{-n/p} \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j(2^{j_0}\cdot)\,\hat{f}]\|_{L^p}^q \right)^{1/q} \\
&= C\lambda^{-n/p} \left\{ \left(\sum_{j=0}^{j_0} + \sum_{j=j_0+1}^{\infty} \right) 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j(2^{j_0}\cdot)\,\hat{f}]\|_{L^p}^q \right\}^{1/q}.
\end{aligned}
$$

For the first term, we see that

$$
\begin{aligned}
&\sum_{j=0}^{j_0} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j(2^{j_0}\cdot)\,\hat{f}]\|_{L^p}^q = \sum_{j=0}^{j_0} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j(2^{j_0}\cdot)\,(\varphi_0+\varphi_1+\varphi_2)\,\hat{f}]\|_{L^p}^q \\
&\le C \sum_{j=0}^{j_0} 2^{jsq} \|\mathcal{F}^{-1}[(\varphi_0+\varphi_1+\varphi_2)\,\hat{f}]\|_{L^p}^q \le C\left(2^{j_0 s}\|f\|_{B_s^{p,q}}\right)^q \le C\left(\lambda^s \|f\|_{B_s^{p,q}}\right)^q.
\end{aligned}
$$

For the second term, we have

$$
\sum_{j=j_0+1}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j(2^{j_0}\cdot)\,\hat{f}]\|_{L^p}^q = \sum_{j=j_0+1}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_{j-j_0}\,\hat{f}]\|_{L^p}^q \le \left(\lambda^s \|f\|_{B_s^{p,q}}\right)^q.
$$

Combining these estimates, we obtain the desired result. □

We are now ready to prove Theorem 1.2 (1) with $(1/p, 1/q) \in I_2^*$.

Proof of Theorem 1.2 (1) with $(1/p,1/q) \in I_2^*$. Let $(1/p,1/q) \in I_2^*$. Then $\nu_1(p,q) = 1/p + 1/q - 1$. If $(1/p,1/q) \in I_2^*$ and $1/p + 1/q = 1$ then $(1/p,1/q) \in I_1^*$, and we have already proved this case in Theorem 1.2 (1) with $(1/p,1/q) \in I_1^*$. Hence, we may assume $1/p+1/q > 1$. Suppose that $B_s^{p,q}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n)$, where $s < n(1/p+1/q-1)$. Then, since $n(1/p+1/q-1) > 0$, we can take $s_0 > 0$ such that $s \le s_0 < n(1/p+1/q-1)$. Let φ be the Gauss function. By Lemma 2.1, we see that $\|\varphi_\lambda\|_{M^{p,q}} \ge C\lambda^{n(1/q-1)}$ for all $\lambda \ge 1$. On the other hand, by Lemma 4.4, we have

$$
\|\varphi_\lambda\|_{B_{s_0}^{p,q}} \le C\lambda^{s_0 - n/p} \|\varphi\|_{B_{s_0}^{p,q}} \quad \text{for all } \lambda \ge 1.
$$

Hence, using $B_{s_0}^{p,q}(\mathbb{R}^n) \hookrightarrow B_s^{p,q}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n)$, we get

$$
C_1\lambda^{n(1/q-1)} \le \|\varphi_\lambda\|_{M^{p,q}} \le C_2 \|\varphi_\lambda\|_{B_{s_0}^{p,q}} \le C_3 \lambda^{s_0-n/p} \|\varphi\|_{B_{s_0}^{p,q}}
$$

for all $\lambda \ge 1$. However, since $s_0 - n/p < n(1/q-1)$, this is contradiction. Therefore, s must satisfy $s \ge n(1/p+1/q-1)$. The proof is complete.

Our next goal is to prove Theorem 1.2 (1) with $(1/p,1/q) \in I_3^*$.

Lemma 4.5. *Let* $1 \le p \le \infty$, $1 \le q < \infty$ *and* $\epsilon > 0$. *Suppose that* $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ *satisfy* $\operatorname{supp}\varphi \subset [-1/8, 1/8]^n$, $\operatorname{supp}\psi \subset [-1/2,1/2]^n$ *and* $\psi = 1$ *on* $[-1/4,1/4]^n$. *For* $j \in \mathbb{Z}_+$, *set*

$$
f^j(t) = 2^{-jn/p} \sum_{\substack{0<|k_j|\le 2^j, \\ j=1,\cdots,n}} |k|^{-n/p-\epsilon}\, e^{ik\cdot t/2^j}\, \Psi(t/2^j - k),
$$

where $\Psi = \mathcal{F}^{-1}\psi$. *Then* $f^j \in M^{p,q}(\mathbb{R}^n)$ *and there exists a constant* $C > 0$ *such that*

$$
\|V_\Phi[(f^j)_{2^j}]\|_{L^{p,q}} \ge C 2^{-jn(2/p-1/q)-j\epsilon} \quad \text{for all } j \in \mathbb{Z}_+,
$$

where $\Phi = \mathcal{F}^{-1}\varphi$.

Proof. Since $f^j \in \mathcal{S}(\mathbb{R}^n)$, we have $f^j \in M^{p,q}(\mathbb{R}^n)$. We consider the second part. Note that $\operatorname{supp} \varphi(\cdot - \xi) \subset \ell + [-1/4, 1/4]^n$ for all $\ell \in \mathbb{Z}^n$ and $\xi \in \ell + [-1/8, 1, 8]^n$. Since $\operatorname{supp} \psi(\cdot - k) \subset k + [-1/2, 1/2]^n$ and $\psi(t-k) = 1$ if $t \in k + [-1/4, 1/4]^n$, it follows that

$$\begin{aligned}
&\|V_\Phi[(f^j)_{2^j}]\|_{L^{p,q}} \ge \Bigg\{ \sum_{\ell \in \mathbb{Z}^n} \int_{\ell + [-1/8,1/8]^n} \\
&\times \Bigg(\int_{\mathbb{R}^n} \Bigg| 2^{-jn/p} \sum_{\substack{0<|k_j| \le 2^j,\\ j=1,\cdots,n}} |k|^{-n/p-\epsilon} \int_{\mathbb{R}^n} e^{ik\cdot t}\, \Psi(t-k)\, \overline{\Phi(t-x)}\, e^{-i\xi\cdot t}\, dt \Bigg|^p dx \Bigg)^{q/p} d\xi \Bigg\}^{1/q} \\
&\ge (2\pi)^{-n} 2^{-jn/p} \Bigg\{ \sum_{\substack{0<|\ell_j| \le 2^j,\\ j=1,\cdots,n}} \int_{\ell + [-1/8,1/8]^n} \\
&\quad \times \Bigg(\int_{\mathbb{R}^n} \Bigg| \sum_{\substack{0<|k_j| \le 2^j,\\ j=1,\cdots,n}} |k|^{-n/p-\epsilon} e^{i|k|^2} \int_{\mathbb{R}^n} e^{-ik\cdot t}\, \psi(t-k)\, \overline{\varphi(t-\xi)}\, e^{ix\cdot t}\, dt \Bigg|^p dx \Bigg)^{q/p} d\xi \Bigg\}^{1/q} \\
&= (2\pi)^{-n} 2^{-jn/p} \Bigg\{ \sum_{\substack{0<|\ell_j| \le 2^j,\\ j=1,\cdots,n}} \int_{\ell + [-1/8,1/8]^n} \\
&\quad \times \Bigg(\int_{\mathbb{R}^n} \Bigg| |\ell|^{-n/p-\epsilon} \int_{\mathbb{R}^n} e^{i(x-\ell)\cdot t}\, \overline{\varphi(t-\xi)}\, dt \Bigg|^p dx \Bigg)^{q/p} d\xi \Bigg\}^{1/q} \\
&= 2^{-jn/p} \Bigg\{ \sum_{\substack{0<|\ell_j| \le 2^j,\\ j=1,\cdots,n}} |\ell|^{-(n/p+\epsilon)q} \int_{\ell + [-1/8,1/8]^n} \|\Phi(-\cdot + \ell)\|_{L^p}^q \, d\xi \Bigg\}^{1/q} \\
&= 4^{-n/q} \|\Phi\|_{L^p} 2^{-jn/p} \Bigg\{ \sum_{\substack{0<|\ell_j| \le 2^j,\\ j=1,\cdots,n}} |\ell|^{-(n/p+\epsilon)q} \Bigg\}^{1/q} \\
&\ge C_n 2^{-jn/p} 2^{-j(n/p+\epsilon)} \Bigg\{ \sum_{\substack{0<|\ell_j| \le 2^j,\\ j=1,\cdots,n}} 1 \Bigg\}^{1/q} \ge C_n 2^{-jn(2/p-1/q)-j\epsilon}
\end{aligned}$$

for all $j \in \mathbb{Z}_+$. The proof is complete. $\square$

Lemma 4.6. *Suppose that $1 \le p, q \le \infty$ and $s > 0$. Let f^j be as in Lemma 4.5. Then there exists a constant $C > 0$ such that $\|(f^j)_{2^j}\|_{B_s^{p,q}} \le C 2^{j(s-n/p)}$ for all $j \in \mathbb{Z}_+$.*

Proof. By Lemma 4.4, we have $\|(f^j)_{2^j}\|_{B_s^{p,q}} \le C 2^{j(s-n/p)} \|f^j\|_{B_s^{p,q}}$ for all $j \in \mathbb{Z}_+$. Hence, it is enough to prove that $\sup_{j \in \mathbb{Z}_+} \|f^j\|_{B_s^{p,q}} < \infty$. Since

$$\widehat{f^j}(\xi) = 2^{jn(1-1/p)} \sum_{\substack{0<|k_j| \le 2^j,\\ j=1,\cdots,n}} |k|^{-n/p-\epsilon}\, e^{-ik\cdot(2^j\xi - k)}\, \psi(2^j\xi - k)$$

and $\operatorname{supp}\psi(2^j\cdot -k)\subset k/2^j+[-2^{-(j+1)},2^{-(j+1)}]^n$, we see that $\operatorname{supp}\widehat{f^j}\subset\{\xi:|\xi|\le 2\sqrt{n}\}$. Let ℓ_0 be such that $2^{\ell_0-1}\ge 2\sqrt{n}$. Then,

$$\|f^j\|_{B^{p,q}_s}=\left(\sum_{\ell=0}^{\ell_0-1}2^{\ell sq}\|\Phi_\ell * f^j\|_{L^p}^q\right)^{1/q}\le\left(\sum_{\ell=0}^{\ell_0-1}2^{\ell sq}(\|\Phi_\ell\|_{L^1}\|f^j\|_{L^p})^q\right)^{1/q}=C_n\|f^j\|_{L^p}.$$

Therefore, it is enough to show that $\sup_{j\in\mathbb{Z}_+}\|f^j\|_{L^p}<\infty$. By a change of variable, we have

$$\begin{aligned}\|f^j\|_{L^p}&=\left(\int_{\mathbb{R}^n}\Bigg|\sum_{\substack{0<|k_j|\le 2^j,\\ j=1,\cdots,n}}|k|^{-n/p-\epsilon}\,e^{ik\cdot t}\,\Psi(t-k)\Bigg|^p dt\right)^{1/p}\\ &\le\left\{\sum_{m\in\mathbb{Z}^n}\int_{m+[-1/2,1/2]^n}\left(\sum_{k\neq 0}|k|^{-n/p-\epsilon}\,|\Psi(t-k)|\right)^p dt\right\}^{1/p}\\ &\le C\left\{\sum_{m\in\mathbb{Z}^n}\left(\sum_{k\neq 0}|k|^{-n/p-\epsilon}\,(1+|m-k|)^{-n-1}\right)^p\right\}^{1/p}<\infty\end{aligned}$$

for all $j\in\mathbb{Z}_+$. The proof is complete. □

We are now ready to prove Theorem 1.2 (1) with $(1/p,1/q)\in I_3^*$.

Proof of Theorem 1.2 (1) with $(1/p,1/q)\in I_3^*$. Let $(1/p,1/q)\in I_3^*$. Then $\nu_1(p,q)=-1/p+1/q$. If $(1/p,1/q)\in I_3^*$ and $p=q$ then $(1/p,1/q)\in I_1^*$, and we have already proved this case in Theorem 1.2 (1) with $(1/p,1/q)\in I_1^*$. Hence, we may assume $1/q>1/p$. Note that $q\neq\infty$. Suppose that $B^{p,q}_s(\mathbb{R}^n)\hookrightarrow M^{p,q}(\mathbb{R}^n)$, where $s<-n(1/p-1/q)$. Then, since $-n(1/p-1/q)>0$, we can take $s_0>0$ such that $s\le s_0<-n(1/p-1/q)$. Set $s_0=-n(1/p-1/q)-\epsilon$, where $\epsilon>0$. For this ϵ, we define f^j by

$$f^j(t)=2^{-jn/p}\sum_{\substack{0<|k_j|\le 2^j,\\ j=1,\cdots,n}}|k|^{-n/p-\epsilon/2}\,e^{ik\cdot t/2^j}\,\Psi(t/2^j-k),$$

where $j\in\mathbb{Z}_+$, $\Psi=\mathcal{F}^{-1}\psi$ and ψ is as in Lemma 4.5. Then, since $B^{p,q}_{s_0}(\mathbb{R}^n)\hookrightarrow B^{p,q}_s(\mathbb{R}^n)\hookrightarrow M^{p,q}(\mathbb{R}^n)$, by Lemmas 4.5 and 4.6, we get

$$\begin{aligned}C_1 2^{-jn(2/p-1/q)-j\epsilon/2}&\le\|V_\Phi[(f^j)_{2^j}]\|_{L^{p,q}}\le C_2\|(f^j)_{2^j}\|_{M^{p,q}}\\ &\le C_3\|(f^j)_{2^j}\|_{B^{p,q}_{s_0}}\le C_4 2^{j(s_0-n/p)}=C_4 2^{-jn(2/p-1/q)-j\epsilon}\end{aligned}$$

for all $j\in\mathbb{Z}_+$, where $\Phi=\mathcal{F}^{-1}\varphi$ and φ is as in Lemma 4.5. However, this is contradiction. Therefore, s must satisfy $s\ge -n(1/p-1/q)$. The proof is complete.

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